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BOUNDARY ELEMENT METHOD SOLUTION OF MHD FLOW IN A RECTANGULAR DUCT

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SUMMARY

The magnetohydrodynamic (MHD) flow of an incompressible, viscous, electrically conducting fluid in a rectangular duct with an external magnetic field applied transverse to the flow has been investigated. The walls parallel to the applied magnetic field are conducting while the other two walls which are perpendicular to the field are insulators. The boundary element method (BEM) with constant elements has been used to cast the problem into the form of an integral equation over the boundary and to obtain a system of algebraic equations for the boundary unknown values only. The solution of this integral equation presents no problem as encountered in the solution of the singular integral equations for interior methods. Computations have been carried out for several values of the Hartmann number ($1 \leqslant M \leqslant 10$). It is found that as M increases, boundary layers are formed close to the insulated boundaries for both the velocity and the induced magnetic field and in the central part their behaviours are uniform. Selected graphs are given showing the behaviours of the velocity and the induced magnetic field.

KEY WORDS BEM MHD flow

1. INTRODUCTION

The study of flows of conducting fluids in ducts in the presence of a transverse magnetic field is important owing to its practical applications in magnetohydrodynamic (MHD) generators, pumps, accelerators and flowmeters. In general, MHD flow problems are extremely complex because of the coupling of the equations of fluid mechanics and electrodynamics, and analytical solutions are out of the question. Various forms of the problem with different combinations of conducting and non-conducting walls have been considered by several authors.[1–5] As pointed out by Hunt,[5] no satisfactory approximate or exact solutions exist for the most important practical case of a rectangular duct with conducting walls parallel to the field and non-conducting walls perpendicular to the field. Grinberg[6,7] has attempted an exact analysis using a Green function method, but his result is incomplete. Later Hunt and Stewartson[8] and Chiang and Lundgren[9] used boundary layer methods to cast the same problem into the form of an integral equation. Singh and Agarwal[10] followed Grinberg's solution procedure for the analytical part but solved the resulting singular integral equation numerically since it could not be solved easily. The finite element method has also been used for solving MHD channel flow problems with different wall conductances.[11,12] Tezer-Sezgin[13] has solved the MHD duct flow problem with a wall which is partly conducting and partly insulating by reducing the problem to the solution of dual series equations.

The present paper uses the boundary element method (BEM) to solve the MHD flow problem in a rectangular duct with perfectly conducting walls parallel to the applied magnetic field and

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non-conducting walls perpendicular to the field. We consider the flow of an incompressible, viscous, electrically conducting fluid in the duct with an external magnetic field applied transverse to the flow. Following the procedure given by Grinberg[6,7] and described in detail by Dragos,[2] the equations are decoupled first for the applicability of the BEM. The BEM with constant elements has been used to cast the problem into the form of an integral equation over the boundary. The unknowns in the resulting discretized system of equations appear only on the boundary points, thus reducing the size of the system of equations. The difficulty arising from the unspecified values of unknown functions and their normal derivatives on the conducting boundary has been overcome by adding the boundary conditions to the resulting system of equations. Thus the BEM solution is a new application for solving the same problem that has been tried in the form of an integral equation in References 6, 7 and 10. Grinberg's solution[6,7] was rigorous from the mathematical point of view, but its practical usefulness was limited since the integral equation could not be solved easily. Our integral equation over the boundary does not present this problem. The singular points were treated separately. Results are obtained for various values of the Hartmann number up to 10 and they compare well with Singh and Agarwal's[10] results.

2. BASIC EQUATIONS

The equations governing the steady, laminar, fully developed flow of an incompressible, viscous, electrically conducting fluid in a rectangular duct subjected to a constant and uniform applied magnetic field are well known and are discussed by Shercliff,[1] Dragos[2] and others. Using a standard non-dimensional form, the governing equations can be written as

$$\nabla^2 V + M \frac{\partial B}{\partial y} = -1 \quad \text{in } \Omega, \tag{1}$$

$$\nabla^2 B + M \frac{\partial V}{\partial y} = 0 \quad \text{in } \Omega, \tag{2}$$

where Ω denotes the section of the duct, $V(y, z)$ and $B(y, z)$ are the velocity and the induced magnetic field respectively and M is the Hartmann number. Here it is assumed that the applied magnetic field B_0 is parallel to the y-axis. $V(y, z)$ and $(B(y, z)$ are in the x-direction, which is the axis of the duct, and the fluid is driven down the duct by means of a constant pressure gradient. The duct walls are at $z = \pm L$ and $y = \pm 1$ (Figure 1). The side walls parallel to the applied magnetic field are perfectly conducting and the horizontal walls perpendicular to the imposed magnetic induction are insulators. Accordingly, the boundary conditions for equations (1) and (2) relating to the configuration of the problem in Figure 1 are

$$V(y, \pm L) = 0, \qquad \left.\frac{\partial B}{\partial z}\right|_{z=\pm L} = 0, \qquad |y| \leqslant 1, \tag{3}$$

$$V(\pm 1, z) = 0, \qquad B(\pm 1, z) = 0, \qquad |z| \leqslant L. \tag{4}$$

Let us also assume that there is no electric current flowing in the direction of the x-axis and thus $B_y = 1$ and $B_z = 0$.

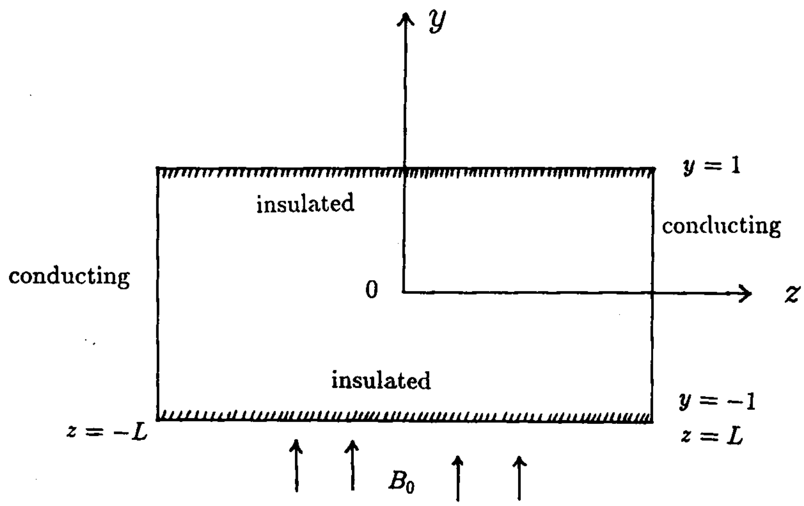

Figure 1. Geometry of the duct

Since both the boundary conditions and the problem equations are invariant with respect to the transformation $z \to -z$, it follows that the solution is even with respect to the variable z, i.e.

$$V(y, z) = V(y, -z), \qquad B(y, z) = B(y, -z), \tag{5}$$

and thus

$$\frac{\partial V}{\partial z}(y, z) = -\frac{\partial V}{\partial z}(y, -z), \qquad \frac{\partial B}{\partial z}(y, z) = -\frac{\partial B}{\partial z}(y, -z). \tag{6}$$

Since the variables are continuous, it follows that

$$\frac{\partial V}{\partial z}(y, 0) = 0, \qquad \frac{\partial B}{\partial z}(y, 0) = 0, \tag{7}$$

so that it is sufficient to solve the problem in the semiduct $0 \leqslant z \leqslant L$.

With the change of variables

$$U_1 = V + B, \qquad U_2 = V - B \tag{8}$$

it follows that

$$\nabla^2 U_1 + M \frac{\partial U_1}{\partial y} = -1, \tag{9}$$

$$\nabla^2 U_2 - M \frac{\partial U_2}{\partial y} = -1, \tag{10}$$

with the boundary conditions

$$\left.\frac{\partial U_1}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial U_1}{\partial z}\right|_{z=L} = f(y), \qquad |y| \leqslant 1, \tag{11}$$

$$U_1(\pm 1, z) = 0, \qquad 0 \leqslant z \leqslant L, \tag{12}$$

$$\left.\frac{\partial U_2}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial U_2}{\partial z}\right|_{z=L} = f(y), \qquad |y| \leqslant 1, \tag{13}$$

$$U_2(\pm 1, z) = 0, \qquad 0 \leqslant z \leqslant L, \tag{14}$$

where $f(y)$ is an unknown function. The first condition in (3) for V becomes

$$U_1(y, L) + U_2(y, L) = 0. \tag{15}$$

The problem is further simplified by the change of variables

$$MU_1 = MW_1 - y, \qquad MU_2 = MW_2 + y, \tag{16}$$

giving the equations

$$\nabla^2 W_1 + M \frac{\partial W_1}{\partial y} = 0, \tag{17}$$

$$\nabla^2 W_2 - M \frac{\partial W_2}{\partial y} = 0. \tag{18}$$

Then by the transformation

$$W_1 = u_1 \mathrm{e}^{-ky}, \qquad W_2 = u_2 \mathrm{e}^{ky}, \tag{19}$$

with $2k = M$, finally one obtains

$$\nabla^2 u_1 - k^2 u_1 = 0, \tag{20}$$

$$\nabla^2 u_1 - k^2 u_2 = 0, \tag{21}$$

with the boundary conditions

$$\left.\frac{\partial u_1}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial u_1}{\partial z}\right|_{z=L} = f(y) e^{ky}, \qquad |y| \leqslant 1, \tag{22}$$

$$2ku_1(\pm 1, z) = \pm e^{\pm k}, \qquad 0 \leqslant z \leqslant L, \tag{23}$$

$$\left.\frac{\partial u_2}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial u_2}{\partial z}\right|_{z=L} = f(y) e^{-ky}, \qquad |y| \leqslant 1, \tag{24}$$

$$2ku_2(\pm 1, z) = \mp e^{\mp k}, \qquad 0 \leqslant z \leqslant L. \tag{25}$$

In order to homogenize the conditions on the boundaries $y = \pm 1$, one should also make the change

$$u_1(y, z) = a(y) + v_1(y, z), \tag{26}$$

$$u_2(y, z) = a(-y) + v_2(y, z), \tag{27}$$

where $a(y)$ is a particular solution of equations (20) and (21) satisfying the condition (23):

$$2ka(y) = \tanh(k)\cosh(ky) + \coth(k)\sinh(ky). \tag{28}$$

With this we get

$$\nabla^2 v_1 - k^2 v_1 = 0, \tag{29}$$

$$\nabla^2 v_2 - k^2 v_2 = 0, \tag{30}$$

with the boundary conditions

$$\left.\frac{\partial v_1}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial v_1}{\partial z}\right|_{z=L} = f(y)e^{ky}, \qquad |y| \leqslant 1, \tag{31}$$

$$v_1(\pm 1, z) = 0, \qquad 0 \leqslant z \leqslant L, \tag{32}$$

$$\left.\frac{\partial v_2}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial v_2}{\partial z}\right|_{z=L} = f(y)e^{-ky}, \qquad |y| \leqslant 1, \tag{33}$$

$$v_2(\pm 1, z) = 0, \qquad 0 \leqslant z \leqslant L, \tag{34}$$

while condition (15) becomes

$$v_1(y, L)e^{-ky} + v_2(y, L)e^{ky} = 2[\cosh^2(ky) - \cosh^2(k)]/k \sinh(2k). \tag{35}$$

In this manner it is sufficient to find the general expression for functions v_1 and v_2 in the semiduct (Figure 2).

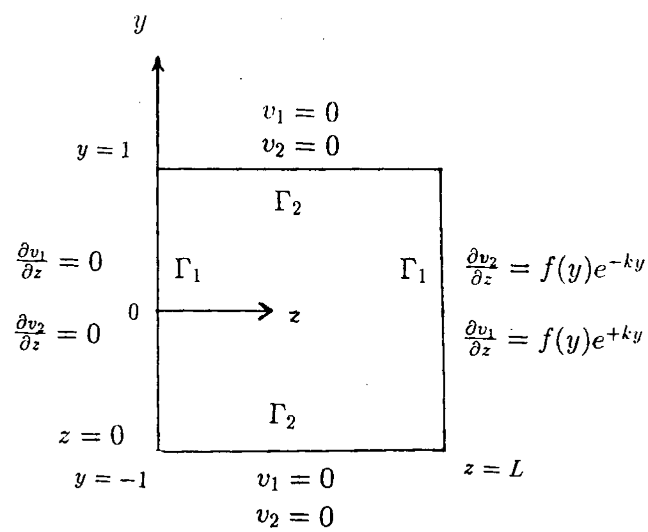

Figure 2. Semiduct and boundary conditions

This problem has already been solved by Dragos[2] and Grinberg[6,7] by reducing it to the solution of an integral equation for the unknown function $f(y)$. Although, as Grindberg says, his solution is rigorous from the mathematical point of view, its practical usefulness is limited since the integral equation cannot be solved easily with its kernel in the double-series form for both small and large Hartmann numbers. Other solutions must be considered. Hence the boundary element method solution has the advantage of having the integral equations for v_1 and v_2 only on the boundary and solutions of these boundary integral equations with this method present no problem for small Hartmann numbers $1 \leqslant M \leqslant 10$. For large Hartmann numbers the resulting system of equations arises with an ill-conditioned coefficient matrix because of the conditions on the boundary $z = L$ (containing terms e^{-ky} and e^{ky}).

3. BOUNDARY ELEMENT FORMULATION

The fundamental motivation behind the boundary element method[14–17] is the reduction of the dimensionality of the problem. Thus we simplify the problem from one involving area integration (e.g. finite element method) to one involving line integration. In general the number of equations derived from such a formulation will be fewer than in the case of an interior method. In contrast with the sparse matrices encountered in other methods, the boundary-element-generated matrices are full. The first step in this formulation involves the generation of an integral equation over the boundary. Thus we need integral representations for $v_1(y, z)$ and $v_2(y, z)$ over the boundary $\partial\Omega$. As we notice from equations (29), (30) and (31)–(34), v_1 and v_2 satisfy the same differential equation and boundary conditions except at $z = L$ (having the same unknown function $f(y)$ but a sign difference in exponential functions). Thus v_1 and v_2 will have the same integral representations over $\partial\Omega$ with the extra conditions (31), (33) and (35) at $z = L$. We use a weighted residual approach because of its inherent simplicity for obtaining boundary integral equations for v_1 and v_2. Introducing a weighting function Ψ which has continuous first derivatives and which satisfies the governing equation (equation (29) for v_1), we can write the weighted residual statement as

$$\int_{\Omega} (\nabla^2 v_1 - k^2 v_1)\Psi \, d\Omega = 0. \tag{36}$$

Employing Green's theorem in two steps to yield

$$\int_{\Omega} (\nabla^2 \Psi - k^2 \Psi) v_1 \, d\Omega + \int_{\partial\Omega} \frac{\partial v_1}{\partial n} \Psi \, dS - \int_{\partial\Omega} v_1 \frac{\partial \Psi}{\partial n} \, dS = 0, \tag{37}$$

we simply have an integral over the boundary $\partial\Omega$, i.e.

$$\int_{\partial\Omega} \frac{\partial v_1}{\partial n} \Psi \, dS - \int_{\partial\Omega} v_1 \frac{\partial \Psi}{\partial n} \, dS = 0, \tag{38}$$

since the weighting function Ψ satisfies the differential equation (29).

We can follow the procedure in References 17 and 18 and select as our weighting function $\Psi = K_0(kr)$ (a modified Bessel function of the second kind and of order zero), the singular solution to the Helmholtz equation. The distance r is measured from an arbitrary point P to a point Q on the boundary. Substituting for Ψ, equation (38) becomes

$$\int_{\partial\Omega} \left(\frac{\partial v_1}{\partial n} K_0(kr) - v_1 \frac{\partial K_0(kr)}{\partial n} \right) dS = 0. \tag{39}$$

The integrals in (39) are readily evaluated except in the vicinity of the singular point P. This point is excluded from the region by a small circle of radius r_0. We now write the required integrations as

$$\int_{\partial\Omega}\left(\frac{\partial v_1}{\partial n}K_0(kr)-v_1\frac{\partial K_0(kr)}{\partial n}\right)\mathrm{d}S+\int_0^{2\pi}\left(\frac{\partial v_1}{\partial n}K_0(kr_0)-v_1\frac{\partial K_0(kr_0)}{\partial n}\right)r_0\,\mathrm{d}\Theta, \tag{40}$$

where Θ increases counterclockwise when the integral along $\partial\Omega$ is taken in a clockwise direction. We now examine (40) in the limit as $r_0 \to 0$ and find that the second term reduces to

$$-\int_0^{2\pi} v_1\frac{\partial K_0(kr_0)}{\partial n}r_0\,\mathrm{d}\Theta=-2\pi v_1(P), \tag{41}$$

since $K_0(kr_0)$ behaves like $-\log(kr_0)$ for small arguments and $\lim_{r_0\to 0} r_0\log(kr_0)=0$, implying

$$\int_0^{2\pi}\frac{\partial v_1}{\partial n}K_0(kr_0)r_0\,\mathrm{d}\Theta=0. \tag{42}$$

The boundary integral equation is now

$$v_1(P)=\frac{1}{2\pi}\int_{\partial\Omega}\left(\frac{\partial v_1}{\partial n}K_0(kr)-v_1\frac{\partial K_0(kr)}{\partial n}\right)\mathrm{d}S. \tag{43}$$

This equation provides a relationship between any point in the interior of Ω and information known only at the boundary. When we require a relationship at the boundary, P is located along the boundary $\partial\Omega$ and equation (39) is written as (for a smooth boundary)

$$\int_{\partial\Omega-\sigma}\left(\frac{\partial v_1}{\partial n}K_0(kr)-v_1\frac{\partial K_0(kr)}{\partial n}\right)\mathrm{d}S+\int_0^{\pi}\left(\frac{\partial v_1}{\partial n}K_0(kr_0)-v_1\frac{\partial K_0(kr_0)}{\partial n}\right)r_0\,\mathrm{d}\Theta=0, \tag{44}$$

where σ is the semicircle around the point P with a radius r_0 which is collinear with the normal.

Taking the limit as $r_0 \to 0$, we have

$$v_1(P)=\frac{1}{\pi}\int_{\partial\Omega}\left(\frac{\partial v_1}{\partial n}K_0(kr)-v_1\frac{\partial K_0(kr)}{\partial n}\right)\mathrm{d}S. \tag{45}$$

Similarly, for differential equation (30) we can obtain integral equations over $\partial\Omega$ as

$$v_2(P)=\frac{1}{2\pi}\int_{\partial\Omega}\left(\frac{\partial v_2}{\partial n}K_0(kr)-v_2\frac{\partial K_0(kr)}{\partial n}\right)\mathrm{d}S \tag{46}$$

when the point P is in Ω and

$$v_2(P)=\frac{1}{\pi}\int_{\partial\Omega}\left(\frac{\partial v_2}{\partial n}K_0(kr)-v_2\frac{\partial K_0(kr)}{\partial n}\right)\mathrm{d}S \tag{47}$$

when P is located along the boundary $\partial\Omega$.

We rewrite equations (45) and (47) in the form

$$v_1(P) = \frac{1}{\pi}\left[\int_{\Gamma_1}\left(\frac{\partial v_1}{\partial n} K_0(kr) - v_1 \frac{\partial K_0(kr)}{\partial n}\right) dS + \int_{\Gamma_2}\left(\frac{\partial v_1}{\partial n} K_0(kr) - v_1 \frac{\partial K_0(kr)}{\partial n}\right) dS\right], \quad (48)$$

$$v_2(P) = \frac{1}{\pi}\left[\int_{\Gamma_1}\left(\frac{\partial v_2}{\partial n} K_0(kr) - v_2 \frac{\partial K_0(kr)}{\partial n}\right) dS + \int_{\Gamma_2}\left(\frac{\partial v_2}{\partial n} K_0(kr) - v_2 \frac{\partial K_0(kr)}{\partial n}\right) dS\right], \quad (49)$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

Now in equations (48) and (49) we notice that on the Γ_2-part of the boundary $\partial\Omega$ both v_1 and v_2 are known and their normal derivatives are unknown. However, on Γ_1, at $z = 0$, $\partial v_1/\partial n$ and $\partial v_2/\partial n$ are known while v_1 and v_2 are unknown; on the other hand, at $z = L$ both pairs v_1, v_2 and $\partial v_1/\partial n$, $\partial v_2/\partial n$ are unknown since $f(y)$ is an unknown function. Therefore we can solve the integral equations (48) and (49) with the help of two extra conditions, namely

$$v_1(y, L)e^{-ky} + v_2(y, L)e^{ky} = 2[\cosh^2(ky) - \cosh^2(k)]/k \sinh(2k), \quad (50)$$

$$\left.\frac{\partial v_1}{\partial n}\right|_{z=L} e^{-ky} - \left.\frac{\partial v_2}{\partial n}\right|_{z=L} e^{ky} = 0, \quad (51)$$

where equation (51) is obtained from the elimination of $f(y)$ at $z = L$ from equations (31) and (33).

4. DEVELOPMENT OF A SET OF SIMULTANEOUS EQUATIONS

The boundary $\partial\Omega$ is discretized into N elements (say N_1 elements on Γ_1 and N_2 elements on Γ_2) and the values of v_1 and v_2 and their normal derivatives are assumed to be constant on each element and equal to the value at the midpoint of the element, i.e.

$$v_1(\xi) = \sum_{j=1}^{N} v_{1,j}\phi_j(\xi), \quad (52)$$

$$v_2(\xi) = \sum_{j=1}^{N} v_{2,j}\phi_j(\xi), \quad (53)$$

$$\frac{\partial v_1}{\partial n}(\xi) = \sum_{j=1}^{N}\left(\frac{\partial v_1}{\partial n}\right)_j \phi_j(\xi), \quad (54)$$

$$\frac{\partial v_2}{\partial n}(\xi) = \sum_{j=1}^{N}\left(\frac{\partial v_2}{\partial n}\right)_j \phi_j(\xi). \quad (55)$$

Here the $\phi(\xi)$'s are suitable interpolation functions and for constant elements

$$\phi_j = \begin{cases} 1, & \xi_j - \Delta\xi_j/2 \leqslant \xi \leqslant \xi_j + \Delta\xi_j/2, \\ 0, & \text{elsewhere,} \end{cases} \quad (56)$$

where ξ is the dimensionless co-ordinate defined over an element and $\Delta\xi_j$ is the length of the jth element (Figure 3). Thus equations (48) and (49) become in discretized form

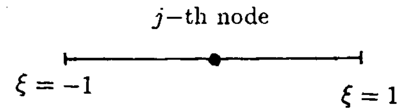

Figure 3. Constant element

$$\pi v_{1,i} = -\sum_{j=1}^{N} v_{1,j} \int_{\Gamma_j} \frac{\partial K_0(kr_i)}{\partial n}\, ds + \sum_{j=1}^{N} \left(\frac{\partial v_1}{\partial n}\right)_j \int_{\Gamma_j} K_0(kr_i)\, ds, \tag{57}$$

$$\pi v_{2,i} = -\sum_{j=1}^{N} v_{2,j} \int_{\Gamma_j} \frac{\partial K_0(kr_i)}{\partial n}\, ds + \sum_{j=1}^{N} \left(\frac{\partial v_2}{\partial n}\right)_j \int_{\Gamma_j} K_0(kr_i)\, ds. \tag{58}$$

Now consider the set of algebraic equations arising out of (57) and (58). Let the integrals for the undetermined parameters be defined as

$$H_{ij} = -\int_{\Gamma_j} \frac{\partial K_0(kr_i)}{\partial n}\, ds \quad (i \neq j), \tag{59}$$

$$H_{ii} = -\int_{\Gamma_i} \frac{\partial K_0(kr_i)}{\partial n}\, ds - \pi, \tag{60}$$

$$G_{ij} = \int_{\Gamma_j} K_0(kr_i)\, ds, \tag{61}$$

r_i being the distance from node i to node j.

Equations (48) and (49) can now be written as

$$\sum_{j=1}^{N} v_{1,j} H_{ij} + \sum_{j=1}^{N} \left(\frac{\partial v_1}{\partial n}\right)_j G_{ij} = 0 \quad (i = 1, 2, \ldots, N), \tag{62}$$

$$\sum_{j=1}^{N} v_{2,j} H_{ij} + \sum_{j=1}^{N} \left(\frac{\partial v_2}{\partial n}\right)_j G_{ij} = 0 \quad (i = 1, 2, \ldots, N) \tag{63}$$

or

$$[H]\{v_1\} + [G]\left\{\frac{\partial v_1}{\partial n}\right\} = 0, \tag{64}$$

$$[H]\{v_2\} + [G]\left\{\frac{\partial v_2}{\partial n}\right\} = 0, \tag{65}$$

where the elements of the matrices $[H]$ and $[G]$ are as defined in (59)–(61). Recall, however, that N_2 values of v_1 and v_2 and $N_1/2$ values of $\partial v_1/\partial n$ and $\partial v_2/\partial n$ (at the $z = 0$ boundary) are known from the boundary conditions. Thus this information can be transferred to the right-hand side of the system (48), (49) and the remaining equations are reordered. Still in this system we have more unknowns than equations, since at the $z = L$ boundary both pairs v_1, v_2 and $\partial v_1/\partial n$,

$\partial v_2/\partial n$ are unknown. Therefore at this boundary the conditions (35) and (51) are discretized and added to the system (64), (65), making it a square system of algebraic equations

$$[A]\{v\} = \{F\}, \tag{66}$$

where the vector $\{v\}$ contains the values of $v_{1,i}$, $v_{2,i}$, $(\partial v_1/\partial n)_i$ and $(\partial v_2/\partial n)_i$ and the vector $\{F\}$ is known from the boundary information and relation (35). One can now solve for $\{v\}$ using direct solution methods. Now that $v_{1,i}$, $v_{2,i}$, $(\partial v_1/\partial n)_i$ and $(\partial v_2/\partial n)_i$ are known at each point along the boundary, one can compute v_1 and v_2 anywhere in the interior. This is achieved by discretizing the general expressions (43) and (46):

$$2\pi v_{1,i} = \sum_{j=1}^{N} v_{1,j} H_{ij} + \sum_{j=1}^{N} \left(\frac{\partial v_1}{\partial n}\right)_j G_{ij}, \tag{67}$$

$$2\pi v_{2,i} = \sum_{j=1}^{N} v_{2,j} H_{ij} + \sum_{j=1}^{N} \left(\frac{\partial v_2}{\partial n}\right)_j G_{ij}; \tag{68}$$

r_i is now measured from the interior point P.

The integrals in the coefficients H_{ij} and G_{ij} of the matrices $[H]$ and $[G]$ respectively will be evaluated numerically. The integrals in H_{ii} and G_{ii} contain the singularities; therefore when the integral contains the point P, a different algorithm is required than when the integral does not contain P.

When the point P lies outside the interval, we have[19]

$$G_{ij} = \int_{\Gamma_j} K_0(kr_i)\,\mathrm{d}s \approx \Delta\xi_j[K_0(kr_i|_{j-1/2}) + 4K_0(kr_i|_j) + K_0(kr_i|_{j+1/2})]/6, \tag{69}$$

where the term $r_i|_j$ refers to the distance between the point P_i and the location s_j on the interval Γ_j (Figure 4). When the segment Γ_i contains P_i, equation (69) is inappropriate and we use the formula[20]

$$G_{ii} = \int_{\Gamma_i} K_0(kr_i)\,\mathrm{d}s \approx 2\sum_{l=0}^{6} (d_l\beta^{2l+1} - \log \beta c_l \beta^{2l+1})/k, \tag{70}$$

where $\beta = (k\Delta\xi_i)/4$ and the values of d_l and c_l are given in Reference 20 (p. 64).

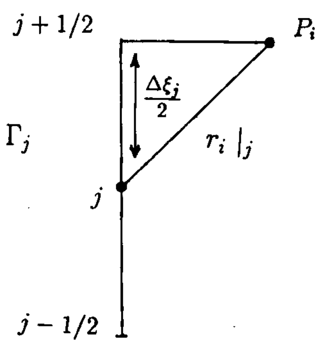

Figure 4. Integration when the point is outside the interval

Because of the slow variation in $\partial K_0(kr_i)/\partial n$ with the position of P, we use the following simple relationship when P lies outside the integration interval:

$$H_{ij} = -\int_{\Gamma_j} \frac{\partial K_0(kr_i)}{\partial n}\, \mathrm{d}s = \int_{\Gamma_j} kK_1(kr_i)\frac{\partial r_i}{\partial n}\, \mathrm{d}s = k\Delta\xi_j K_1(kr_i)\frac{\partial r_i}{\partial n}, \tag{71}$$

where $K_1(x)$ is a modified Bessel function of the second kind and of order one and

$$\frac{\partial r_i}{\partial n} = \frac{(z_j - z_i)\,\partial z_i/\partial n + (y_j - y_i)\,\partial y_j/\partial n}{\sqrt{[(z_i - z_j)^2 + (y_i - y_j)^2]}}, \tag{72}$$

with $(z_i, y_i) = P_i$ and $(z_j, y_j) = P_j$.

When P lies on the integration interval, the integral

$$\int_{\Gamma_i} \frac{\partial K_0(kr_i)}{\partial n}\, \mathrm{d}s$$

is zero since the direction cosine is zero, so

$$H_{ii} = -\pi. \tag{73}$$

Thus when the system (66) with the entries (69)–(71) and (73) in the matrices $[G]$ and $[H]$ is solved for the unknown vector $\{v\}$ which contains the values of v_1, v_2, $\partial v_1/\partial n$ and $\partial v_2/\partial n$ on the boundary nodes, one can easily compute v_1 and v_2 anywhere inside the region through equations (67) and (68). To find the velocity $V(y, z)$ and the induced magnetic field $B(y, z)$ at these interior points, we go back through the relationships (26), (27), (19), (16) and (8). For solving the system of linear algebraic equations (66), Gauss elimination with complete pivoting (the L2ARG matrix solver from the IMSL library) was used.

5. NUMERICAL RESULTS AND DISCUSSION

We have taken a square duct in our calculations (i.e. $L = 1$). It was divided into a mesh by taking mesh sizes of 0·2 in each direction for inside calculations ($-0{\cdot}9 \leqslant y \leqslant 0{\cdot}9$, $0{\cdot}1 \leqslant z \leqslant 0{\cdot}9$). Throughout the computations double precision was used and the Bessel functions K_0 and K_1 were computed using subroutines from the IMSL library in double precision.

The calculations have been done for small values of the Hartmann number, $1 \leqslant M \leqslant 10$. Figures 5–7 show the behaviour of the velocity for several values of M at $z = 0{\cdot}1$, $z = 0{\cdot}9$ and $y = 0{\cdot}1$ respectively. As the Harmann number is increased, the velocity profile shows a flattening tendency along both the y- and the z-axis. Similar behaviour can be seen in the induced magnetic field profiles of Figures 8 and 9 at $z = 0{\cdot}1$ and $z = 0{\cdot}9$ respectively. Sharp changes in both the velocity and the induced magnetic field are seen near the boundaries $z = \mp 1$ and $y = \mp 1$, implying boundary layers close to these boundaries as M increases. Figures 10 and 11 show the velocity and induced magnetic field contours respectively for $M = 1$ in the semiduct $0 \leqslant z \leqslant 1$, $-1 \leqslant y \leqslant 1$ using symmetry with respect to the z-axis. These curves have been drawn using the MATLAB package and the non-smoothness of the curves is due to the linear interpolation used in that package.

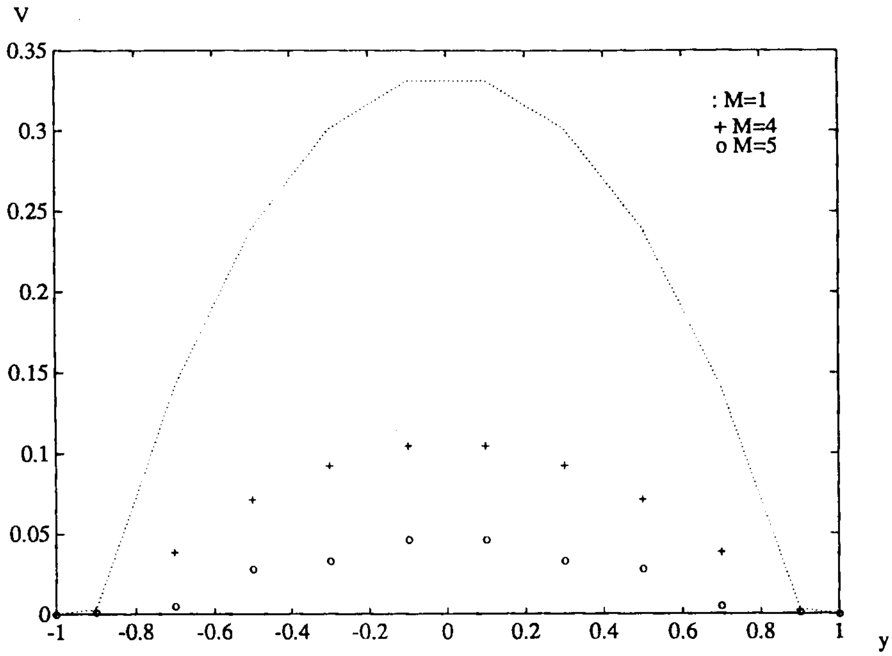

Figure 5. Velocity at $z = 0{\cdot}1$

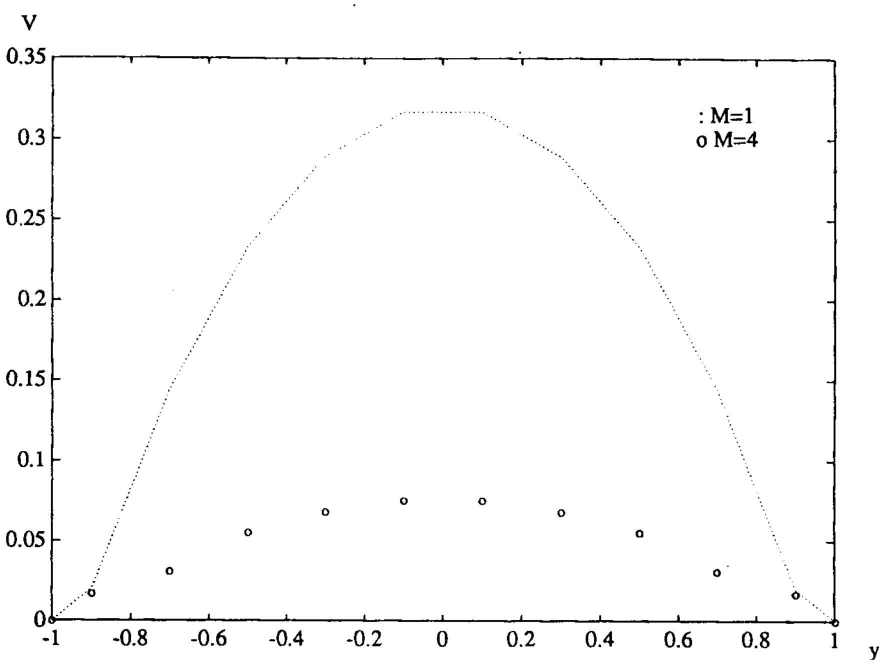

Figure 6. Velocity at $z = 0{\cdot}9$

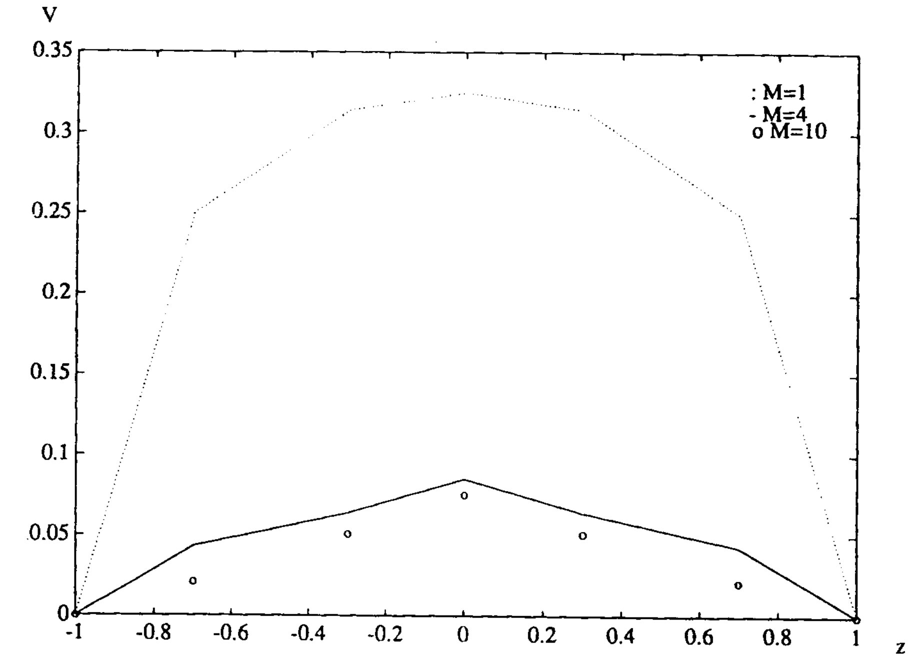

Figure 7. Velocity at $y = 0{\cdot}1$

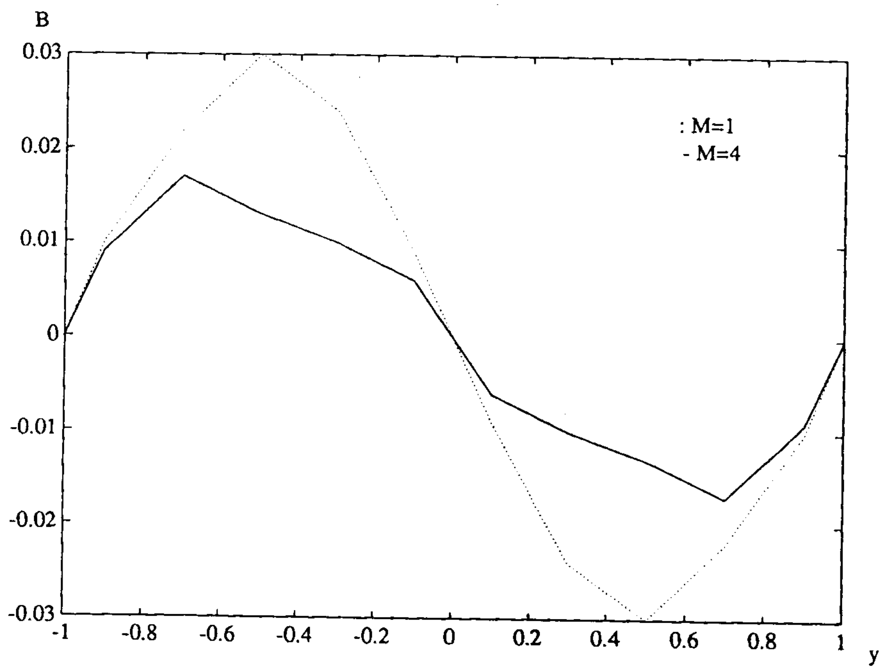

Figure 8. Induced magnetic field at $z = 0{\cdot}1$

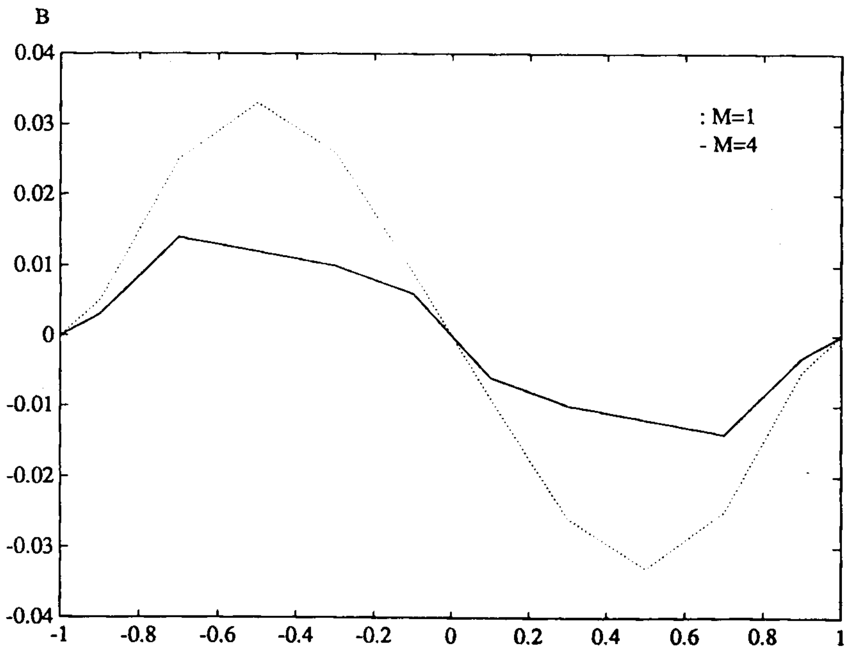

Figure 9. Induced magnetic field at $z = 0{\cdot}9$

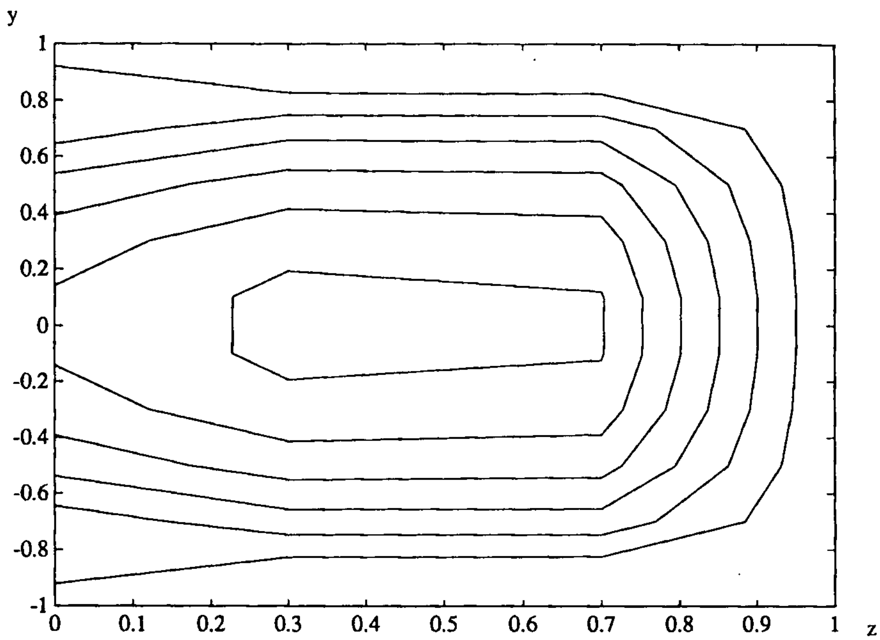

Figure 10. Velocity for $M = 1$

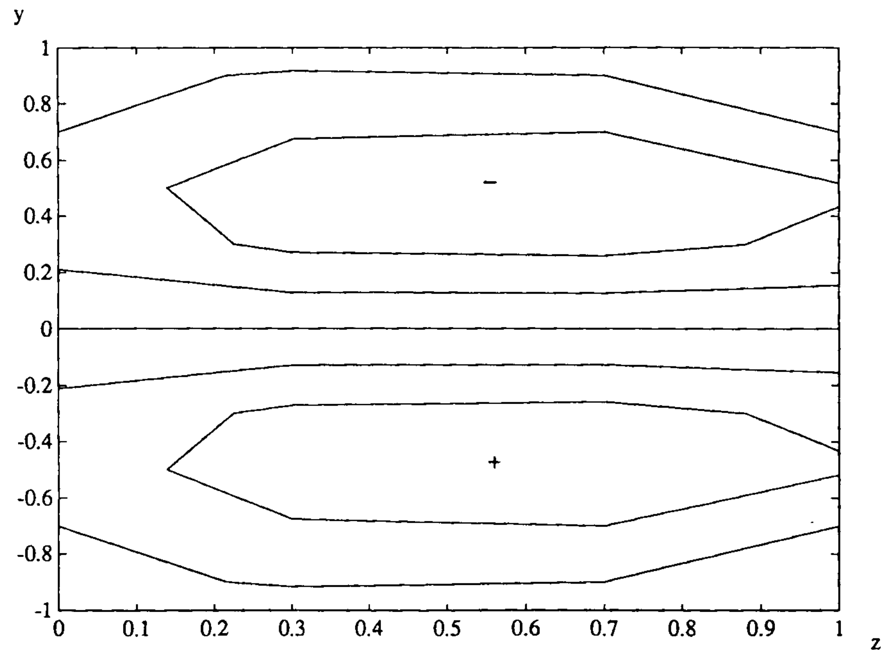

Figure 11. Induced magnetic field for $M = 1$

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